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TECHNICAL NOTES

Convective heat transfer for steady laminar flow between two confocal elliptic pipes with longitudinal uniform wall temperature gradient and uniform heat generation

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INTRODUCTION

CONVECTIVE heat transfer in steady laminar flow for various geometries have been extensively covered in the existing literature. Kays and Crawford [1], Bejan [2], Eckert *et al.* [3] and Shah and London [4] discussed the majority of the more important cases and have given the heat transfer results. However, the one important case of heat transfer between two confocal elliptic pipes with various wall heating conditions is not included in these studies. Besides the scientific interest of this case, the design of a heat exchanger in a narrow space may require information on convective heat transfer in annular elliptic pipes. Additionally, the limiting case of an elliptic pipe with a flat core, heated or cooled independently on its internal and external walls, may also find useful engineering applications [5].

The problem analyzed here is the extension to ref. [5] where uniform heat generation is included. The results for an ellipticity of 0.5 are given graphically to illustrate the heat transfer characteristics of the problem.

Consider an elliptic annular pipe subjected to two independent axially uniform heat fluxes through its inner and outer walls. Along the length of the pipe where the velocity and temperature distribution are fully developed, the temperature distributions must have the functional form, equation (3.6) of ref. [5]

$$\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial \eta^2}\right)e = -h(x_2-2m^2\cos 2\eta) + G(E_0-E_2\cos 2\eta+E_4\cos 4\eta). \quad (1)$$

The solution of equation (1) is given as equation (3.9) in ref. [5]

$$e = e_{\text{in}}g_0+h(f_0+f_2\cos 2\eta)+G(e_0+e_2\cos 2\eta+e_4\cos 4\eta) \quad (2)$$

where the same nomenclature of ref. [5] is retained.

HEAT FLUXES THROUGH THE WALLS

The element of heat flux, du, measured in the positive direction of ξ through an elemental area of ξ = constant cylindrical surface is

$$\mathrm{d}u = -LF\xi\frac{\partial e}{\partial \xi}\mathrm{d}\eta \quad (3)$$

where

$$L = \tfrac{1}{2}(A+B). \quad (4)$$

A and B are the semi-major and semi-minor axes of the cross-section of the outer elliptic pipe, respectively, and F is the characteristic heat flux.

The heat gain rates U_i and U_o, per unit length of inner and outer pipes, respectively, which are taken to be positive when heat flows into the fluid, are expressed as

$$U_i = -2\pi LF\left[\frac{\beta G+\omega hf'_0(\omega)\log\omega+\omega Ge'_0(\omega)\log\omega}{\log\omega}\right] \quad (5)$$

$$U_o = 2\pi LF\left[\frac{\beta G+hf'_0(1)\log\omega+Ge'_0(1)\log\omega}{\log\omega}\right] \quad (6)$$

with

$$e'_0(\omega) = \frac{1}{\omega}[e'_0(1)-I_{00}] \quad (7)$$

and

$$e'_0(1) = \tfrac{1}{4}(1-m^8)-\frac{1-\omega^2}{1+\omega^2}m^4-\tfrac{1}{4}u_1(1+m^4) -\tfrac{3}{16}u_1(1+\omega^2)\left(1+\frac{m^4}{\omega^2}\right)+\tfrac{1}{4}u_1^2a_1 \quad (8)$$

where $e'_0(\omega)$ and $e'_0(1)$ are the derivatives of e with respect to ξ evaluated at $\xi = \omega$ and 1 and

$$\beta = \frac{E_{\text{in}}}{LC\,Pe} \quad (9)$$

which is an alternate definition for the dimensionless inner wall excess temperature.

Two special values of β are given below.

For an insulated outer wall ($U_o = 0$)

$$\beta = \beta_i = -\frac{hf'_0(1)\log\omega+Ge'_0(1)\log\omega}{G} \quad (10)$$

where

$$f'_0(1) = \tfrac{1}{4}[u_1-2(1-m^4)]. \quad (11)$$

For an insulated inner wall ($U_i = 0$)

$$\beta = \beta_o = -\frac{\omega hf'_0(\omega)\log\omega+\omega Ge'_0(\omega)\log\omega}{G} \quad (12)$$

where

$$f'_0(\omega) = \frac{1}{4\omega}\left[u_1 - 2\left(\omega^2 - \frac{m^4}{\omega^2}\right)\right]. \quad (13)$$

The difference between these being

$$\beta_i - \beta_o = \frac{h\log\omega}{2G}(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right) - I_{00}\log\omega \quad (14)$$

where

$$I_{00} = \tfrac{1}{4}(1-\omega^4)\left(1+\frac{m^8}{\omega^4}\right) - 2m^4\frac{(1-\omega^2)}{(1+\omega^2)} - \tfrac{1}{4}(1-\omega^2)\left(1-\frac{m^4}{\omega^2}\right)u_1. \quad (15)$$

The ratio, λ, of the heat gains from the outer wall to that from both walls per unit length of pipe is obtained as

$$\lambda = \frac{U_o}{U_o + U_i} = \frac{\beta - \beta_i}{\beta_o - \beta_i} \quad (16)$$

depending on the dimensionless inner wall temperature β only.

For the special case of equal wall temperatures, $T_o = T_i(\beta = 0)$, the ratio λ_0 is

$$\lambda_0 = \frac{\tfrac{1}{2}h\left[\left(1-\frac{m^4}{\omega^2}\right)(1-\omega^2) + 2(1-m^4)\log\omega\right] - 2Ge'_0(1)\log\omega}{h\log\omega(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right) - 2GI_{00}\log\omega}. \quad (17)$$

Introducing the ratio, $\mu = \lambda/\lambda_o$, the alternate dimensionless inner wall excess temperature, β, can also be expressed as

$$\beta = -(1-\mu)\frac{hf'_0(1)\log\omega + Ge'_0(1)\log\omega}{G}. \quad (18)$$

After substituting this into equations (5) and (6), the inner and outer heat fluxes are reduced to

$$U_i = 2\pi LF\left\{-\tfrac{1}{2}h(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right) + I_{00}G - \mu[hf'_0(\omega) + Ge'_0(\omega)]\right\} \quad (19)$$

$$U_o = 2\pi LF\{\mu[gf'_0(1) + Ge'_0(1)]\}. \quad (20)$$

The values of λ, β, and μ for three special cases are as follows:

for insulated outer wall ($U_o = 0$)

$$\lambda = 0,\quad \beta = \beta_i,\quad \mu = 0;$$

for equal wall temperatures ($T_o = T_i$)

$$\lambda = \lambda_o,\quad \beta = 0,\quad \mu = 1; \quad (21)$$

for insulated inner wall ($U_i = 0$)

$$\lambda = 1,\quad \beta = \beta_o,\quad \mu = \mu_o = \frac{1}{\lambda_o}.$$

For the cases of equidirectional heat fluxes through both walls, μ must fall into the following ranges:

$$\text{for } \lambda > 1 \text{ and } \mu > \mu_o,\quad -1 < \frac{U_i}{U_o} < 0$$
$$\text{for } \lambda < 0 \text{ and } \mu = -n^2\mu_o,\quad \frac{U_i}{U_o} < -1 \quad (22)$$

where n is any positive factor.

Now it is clear that any possible combinations of the heat fluxes through the walls can be represented by either of the parameters λ or μ.

CONVECTIVE HEAT TRANSFER COEFFICIENTS

The mixed mean bulk temperature, T_m, at any station Z is defined by

$$T_m = \frac{\rho}{Q}\int_S WT\,ds \quad (23)$$

where S and ds are full and elemental cross-sectional areas, respectively. A mixed mean excess temperature, E_m, can be similarly defined as

$$E_m = T_m - T_o = \frac{\rho}{Q}\int_S WE\,ds \quad (24)$$

where

$$Q = 2\pi LRE_L I_{00}. \quad (25)$$

Substituting the above equation for Q into equation (24)

$$E_m = -\frac{FL}{k}\frac{J}{I_{00}} \quad (26)$$

where

$$J = -\frac{1}{2\pi RE_L}\int_0^{2\pi}\int_\omega^1 \xi\left(1+\frac{m^4}{\xi^4} - 2\frac{m^2}{\xi^2}\cos 2\eta\right)we\,d\xi\,d\eta \quad (27)$$

noting that for non-circular cross-sections, peripherally uniform temperature distributions do not correspond to uniform heat flux distributions around the peripheries. The mean convective heat transfer coefficients h_o and h_i for the outer and inner walls may be defined as

$$U_o = (T_o - T_m)P_o h_o,\quad U_i = (T_i - T_m)P_i h_i. \quad (28)$$

Also by using equations (9) and (26), the factor in equation $(28)_2$ can be changed to

$$(E_m - E_{in}) = -\frac{FL}{k}\left(\frac{J}{I_{00}} + G\beta\right). \quad (29)$$

Equations (19), (20) and (28) yield the Nusselt numbers as

$$Nu_o = (DE)_1\cdot\frac{I_{00}}{J}\mu[hf'_0(1) + Ge'_0(1)]$$

$$Nu_i = (DE)_\omega\cdot\frac{I_{00}}{G\beta I_{00} + J}\left\{-h\left(\frac{1-\omega^2}{2}\right)\left(1+\frac{m^4}{2}\right)_\omega + I_{00}G - \mu[hf'_0(1) + Ge'_0(1)]\right\}. \quad (30)$$

The integral J in this case is calculated as

$$J = \beta GY + G(J_0 + J_2 + J_4) + h(J_6 + J_8) \quad (31)$$

where

$$Y = -\int_\omega^1 E_0 g\frac{1}{\xi}d\xi,\quad J_0 = -\int_\omega^1 E_0 e_0\frac{1}{\xi}d\xi \quad (32)$$

$$J_2 = \frac{1}{2}\int_\omega^1 E_2 e_2\frac{1}{\xi}d\xi,\quad J_4 = -\frac{1}{2}\int_\omega^1 E_4 e_4\frac{1}{\xi}d\xi \quad (33)$$

$$J_6 = -\int_\omega^1 E_0 f_0\frac{1}{\xi}d\xi,\quad J_8 = \frac{1}{2}\int_\omega^1 E_2 f_2\frac{1}{\xi}d\xi \quad (34)$$

$$\beta Y = (1-\mu)I_{01}\left[\frac{hf'_0(1) + Ge'_0(1)}{G}\right]$$

$$\text{and}\quad I_{01} = \int_\omega^1 (E_0\log\xi)\frac{1}{\xi}d\xi. \quad (35)$$

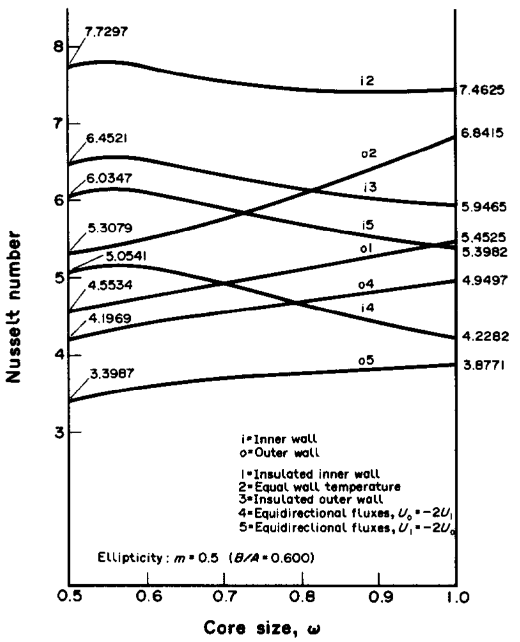

FIG. 1. Nusselt number vs core size for ellipticity $m = 0.5$.

The denominators of Nu_o and Nu_i, equations (30), are also obtained as

$$\mathrm{DENO} = (1-\mu)I_{01}[hf'_0(1)+Ge'_0(1)]+G(J_0+J_2+J_4) + h(J_6+J_8) \quad (36)$$

$$\mathrm{DENI} = \mathrm{DENO}-(1-\mu)\left[\frac{hf'_0(1)\log\omega+Ge'_0(1)\log\omega}{G}\right]I_{00}. \quad (37)$$

Therefore, the Nusselt numbers from equations (30) can be written in their final form as

$$Nu_i = (DE)_\omega \frac{I_{00}}{\mathrm{DENI}}\left\{-h\left(\frac{1-\omega^2}{2}\right)\left(1+\frac{m^4}{\omega^2}\right) + I_{00}G-\mu[hf'_0(1)+Ge'_0(1)]\right\} \quad (38)$$

$$Nu_o = (DE)_1 \frac{\mu}{\mathrm{DENO}} I_{00}[hf'_0(1)+Ge'_0(1)].$$

The values of the parameter μ depend on the desired heating and cooling combinations on the walls. For three special cases, insulated inner wall, insulated outer wall and equal wall temperatures, its values are given in equations (21). For calculation purposes, two additional special values were selected as

$$\lambda = 2: \quad \mu = \frac{2}{\lambda_o}\,\frac{U_i}{U_o} = -\frac{1}{2}$$
$$\lambda = -1: \quad \mu = \frac{1}{\lambda_o}\,\frac{U_i}{U_o} = -2 \quad (39)$$

which represent heating and cooling conditions where the wall heat fluxes are equidirectional but of opposite sign.

With these five different values of μ, the numerical values of the outer and inner Nusselt numbers were calculated over the range of values of m and ω. The results are plotted in Fig. 1 for one value of the ellipticity, $m = 0.5$.

NUMERICAL RESULTS AND CONCLUSIONS

Convective heat transfer in the annulus of two confocal elliptic pipes is analyzed and the results are presented in analytical closed forms. Equations (38) show that the Nusselt numbers depend on five independent parameters, namely: the magnitude of heat generation, the ellipticity of the pipe (the core size), the Reynolds number, the Prandtl number, and the wall temperature gradient.

The Nusselt numbers in equations (38) are plotted vs the core size of the pipe, ω, for values of Prandtl number ($Pr = 0.005$), Reynolds number ($Re_L = 1500$), temperature gradient along the wall ($c = \partial T/\partial Z = 538°\mathrm{C}$), and the magnitude of heat generation ($h = 966\,230\ \mathrm{W\ m^{-3}}$) for various heating combinations as defined in Fig. 1 of ref. [5]. The Prandtl numbers, temperature gradient and magnitude of heat generation are for nuclear coolant (liquid sodium) which has the widest application in nuclear engineering [6].

Nusselt numbers are plotted vs core size, ω, for constant Reynolds number, constant Prandtl number and constant heat generation and for one ellipticity ($m = 0.5$). This is presented in Fig. 1. It can be observed that the Nusselt numbers for the inner pipe initially increases from a core size of $\omega = 0.1$ to 0.18 and then decreases gradually as ellipticity increases. One can conclude that some optimum heat transfer properties exist. For some engineering applications, design of heat exchanger in narrow spaces, these optima may be useful. The Nusselt numbers for the outer pipe will decrease gradually as ellipticity increases. It is seen from Fig. 1 that, in the limiting case of $\omega = 1.0$, which is corresponding to a simple pipe, the outer Nusselt numbers for all possible heating combinations are increasing with the amount of heat generation considered by 14% [5]. Also the inner Nusselt numbers for all possible heating combinations increases with the amount of heat generation considered by 17% [5]. In all cases, however, the heat transfer is augmented for this case with heat generation as that given in ref. [5].

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